

Block Lecture 1: Zero-cycles on Surfaces / c. 11

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Variety: ^{reduced} $\text{irred } q\text{-proj.}$ Scheme: $q\text{-proj.} / \mathbb{C}$ (unless _{other})

Intersection Theory (ground-rules)

X smooth variety. $\mathbb{Z}^r(X) = \text{free abelian on codim. } r \text{ points.}$

Equivalence subgroups

$\mathbb{Z}^r_{\text{Rational}}$

$\mathbb{Z}^r_{\text{Algebraic}}$

$\mathbb{C}$

$\mathbb{Z}^r_{\text{Homological}}$

$\mathbb{C} \mathbb{Z}^r$

$\langle a - b \neq i_* [\text{div } f], \text{ \& meas. on some irred. } C \in \mathbb{Z}^{r+1} \rangle$

Examples: ^{2-cycles} a $\sim$ b by chain of rational curves.

Divisors: linear equiv.
(Equiv. to Bloch's def'n:
Fulton §1.6)

$\langle \pi_* [\pi^* (X \times C) - (b)] \rangle$
 $T \in \mathbb{Z}^r(X \times C)$,
 C smooth curve

homologous as cycles.

Finest equivalence to have intersection products. (Chow ring.)

$$CH^d(X) = \mathbb{Z}^d / \mathbb{Z}^d_{\text{rat}} \supset A^d(X) = \mathbb{Z}^d_{\text{alg}} / \mathbb{Z}^d_{\text{rat}}$$

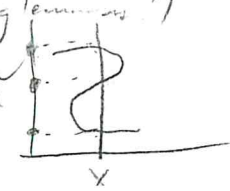
$$CH_d = CH^{n-d}$$

$$A_d = A^{n-d}$$

$CH_0 \supset A_0 = \text{group of degree 0 0-cycles (points).}$

Have flat pullback, finite pushforward ^{or proper}

$\Rightarrow [\pi] \in CH^k(X \times Y)$ gives a correspondence (Fulton - moving lemma?)
 $[\pi]_* : CH^d(X) \xrightarrow{\text{homo.}} CH^{d+k}(Y)$
e.g. $[\pi]_* = f_*$
 $d \mapsto \pi_* (\pi^*(d) \cdot \pi^*)$



Today: X smooth projective surface / $\mathbb{C}$. 12

$$q = H^1(X, \mathcal{O}_X) = H^0(X, \Omega_X^1), \quad p_g = H^2(X, \mathcal{O}_X) = H^0(X, \Omega_X^2)$$

Recall the Albanese variety = higher-dim Jacobian: (Voisin, Beauville)

$$Alb(X) = H^0(X, \Omega_X^1)^* / H_1(X, \mathbb{Z}),$$

well-defined due to closedness of 1-forms / Stokes
(i.e. $\int_{\partial C} \omega = 0$)

$Alb(X)$ is an Abelian variety with polarization coming from cup product form
 $(\eta, \eta') \mapsto i \int_X \eta \wedge \bar{\eta}' \omega$ on $H^0(X, \Omega_X^1)$
 / Hodge theory /aton

Fixing a basepoint $p_0 \in X$, define the Albanese map

$$\phi: p \mapsto \int_{p_0}^p \in Alb.$$

$$H^0(X) \ni \sum p_i \mapsto \sum \int_{p_0}^{p_i}$$

Prop'n

(0) Alb is functorial $X \rightarrow Y$ using dual to pullback of forms.

$$\begin{array}{ccc} X & \xrightarrow{\gamma} & Y \\ \downarrow & \searrow \cong & \downarrow \\ Alb(X) & \xrightarrow{\gamma_*} & Alb(Y) \end{array}$$

$\therefore Im(\phi)$ is not const in a proper sub-torus.

(1) $\phi_n: X \times \dots \times X \rightarrow Alb(X)$ is surjective (look at tangent-spaces, use (0))

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Alb(X) \\ \downarrow & \swarrow & \\ T & = & \mathbb{C}\text{-tors} \end{array} \quad (d\phi_p = \omega(p))$$

(2) ϕ factors $CH_0(X) \rightarrow Alb(X)$. In fact $A_0(X) \twoheadrightarrow Alb(X)$
 (same argument.)

To prove (3): Prop: (intuitive/Fulton 1.6.3)

Rational equivalence is generated by $\{A - B = f(0) - f(\infty) \mid f: \mathbb{P}^1 \rightarrow S^n X\}$ some n .
 $\Rightarrow \mathbb{P}^1 \rightarrow S^n X \rightarrow Alb(X) = \text{const}$ dual to H^1 -forms on torus $T = V^*$

Example 1: $F \times E$, elliptic curves

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$$0 \neq \eta \in E \text{ s.t. } 2\eta = 0$$

$$\text{Let } \sigma(f, e) = (-f, e + \eta)$$

$$\text{Let } E' = E / \{1, \eta\}, \text{ elliptic}$$

$$\text{Have map } X = F \times E / \{1, \sigma\}, \text{ smooth since } \sigma \text{ free.}$$

$$f: X \rightarrow E' \text{ with fibers } \cong F.$$

$$T(X, \mathcal{L}'_X) \cong T(F \times E, \mathcal{L}'_{F \times E}) \quad \text{b/c}$$

$$\text{Kumath} \cong [T(E, \mathcal{L}'_E) \oplus T(F, \mathcal{L}'_F)] \xrightarrow{F \times E \rightarrow X} \text{étale.}$$

$$\cong T(E, \mathcal{L}'_E) \cong C$$

$$\Rightarrow g(X) = 1 = \dim \text{Alb}(X) \text{ since } \sigma = -1 \text{ on } F \text{ on } F.$$

$$\text{Since } X \rightarrow \text{Alb}(X)$$

$$\text{connected fibers} = F \downarrow \swarrow \text{map of curves} \\ E'$$

$$\Rightarrow \text{Alb}(X) \cong E'$$

(X smooth variety)

Lemma 1: $A_n(X)$ is divisible.

(alg. equiv. mod out by any dimension)

Pf: Generated by images

$$C \times X$$

$$\xrightarrow{\pi} T$$

$$\Rightarrow [T]_x: A_0(C) \rightarrow A_n(X)$$

$$\parallel$$

$$J_C \Rightarrow \text{divisible.}$$

Lemma 2: Let

$$T(X) = \ker(A_0(X) \rightarrow \text{Alb}(X))$$

Also divisible.

Example 1 (bis)

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Claim: $A_0(X) \cong E' \cong \text{Alb}(X)$, i.e. $T(X) = 0$.

Proof:

$$\begin{array}{ccc} F \times E & \longrightarrow & E \\ \downarrow \pi & & \downarrow \alpha \\ X & \longrightarrow & E' \end{array}$$

Let $z \in T(X)$, and write

$$\pi^* z = \sum v_i [(q_i, p_i) + (-q_i, p_i + 4)]$$

where $\sum v_i = 0$ b/c $z \in A_0(X)$

and

$$\sum v_i \text{Im}(p_i) = 0 \text{ in } E' \Rightarrow \sum \sum v_i p_i = 0 \text{ in } E.$$

Using Abel's theorem on each component of $F \times E$, (E is its own Jacobian, same for F), we get

$$\begin{aligned} z(q, p) &\sim z(q, p + 4) && \text{on } F \times E \\ (q, p) + (-q, p) &\sim z(0, p) \end{aligned}$$

$$\begin{aligned} \Rightarrow z \pi^* z &\sim \sum \sum v_i [(q_i, p_i) + (-q_i, p_i)] \sim \sum 4(0, p_i) \\ &= [(0) \times E] \cdot \int F \times z \cdot \sum v_i (p_i) \end{aligned}$$

intersection preserves
rational equivalence!
(also corresponds)

~ 0

~ 0

$$\text{Since } \pi_* (z \pi^* z) = 4z \sim 0$$

$\Rightarrow z \sim 0$ since $T(X)$ divisible

□

Example 2: Godeaux Surface:

$$Y: T_0^5 + \dots + T_3^5 = 0 \text{ in } \mathbb{P}^3$$

$$\pi \downarrow X = Y / \mathbb{Z}_5 = \langle (T_0, \omega T_1, \omega^2 T_2, \omega^3 T_3) \rangle, \omega^5 = 1.$$

Get $P_g = 0, q = 0$ ($Y \rightarrow X$ étale, use Noether

$$1_g = 4, q = 0, K^2 = 4^2 = 5 \Rightarrow g(X) = 0$$

$$\chi(Y) = 1 + 4 = 5 \Rightarrow \chi(X) = 1 \Rightarrow \boxed{P_g = 0.}$$

Since π étale,

St: "general type."

$$S A_0(X) = \pi_* \pi^* A_0(X), \text{ so suffices to show } = 0 \text{ by divs.}$$

Since π_* surjective, suff. to show

$$\pi^* \pi_* : A_0(Y) \rightarrow A_0(Y) = 0.$$

Consider $e_i \in \text{Aut}(Y), i = 1, 2, 3, \text{ s.t.}$

$$e_i^* T_j = \begin{cases} T_j & i \neq j \\ \omega T_j & i = j \end{cases}$$

Gives $\mathbb{Z}$ -mod. rep'n $\mathbb{Z}[\mathbb{Z}/5^{\oplus 3}] \rightarrow R \subset \text{End}(A_0(Y))$.

Here, $\pi^* \pi_* = 1 + \bar{e}_1 \bar{e}_2 \bar{e}_3 + \dots$, so we must show this is zero. There are lots of relations in R :

the quotient of Y by $\langle (e_1, e_2, e_3)^i \rangle$ is rational, b/c

have well-defined ^{holo. map} from $\mathbb{P}^3$, $[\sqrt[5]{\sum T_i^2}, T_1, T_2, T_3]$

a rational surface has $A_0 = (0)$, so we conclude

$$0 = \sum (\bar{e}_1 \bar{e}_2 \bar{e}_3)^i \text{ in } R$$

$$\text{Also } \sum_i \bar{e}_k^i = 0 = \sum_k (\bar{e}_i \bar{e}_j)^k, \text{ b/c quotients are rational}$$

~~in contrast to Fermat surface itself, which is general type so $p \neq q \Leftrightarrow p = q$.~~

Example 2 cont'd: Since $A_0(Y)$ divisible,

we can prove in $R \otimes Q = \text{quotient of } \mathbb{Z}[x_1, x_2, x_3] \otimes Q$,

which is f.d. domain/ $Q \Rightarrow$ field $\Rightarrow$ so is $R \otimes Q$.

$\Rightarrow$ has an embedding to $\bar{Q}$! Determined by

images of $e_1, e_2, e_3 \mapsto w_1, w_2, w_3 \in \bar{Q}$ s.t. $w_i^5 = 1$

To show $\pi^* \pi_* = 0$, can just show it goes to

zero under any map $R \otimes Q \rightarrow \bar{Q}$. If not, then

need $w_1 w_2^2 w_3^3 = 1$ (so image of $\pi^* \pi_* = 5$.)

But for the ^{above} identities in R to be repeated,

also need $w_i \neq 1$, $w_i w_j \neq 1$

$w_1 w_2 w_3 \neq 1$

(so that $\Sigma = 0$.)

These are impossible.

$\Rightarrow \pi^* \pi_* = 0$.

Example 3: Fans surface: $P_g > 0$

$S = \{ \text{Lines on cubic } \mathbb{CP}^3 \}$

$\parallel$

Typical

So 0-cycles are lines

divisors on $S =$ set of lines incident on a fixed line.

Block shows:

$$\text{Pic } S \otimes_{\mathbb{Z}} \text{Pic}^0(S) \longrightarrow A_0(S)$$

$$\text{Pic}^0 S \otimes_{\mathbb{Z}} \text{Pic}^0(S) \longrightarrow T(S).$$

So intersections are surjective!!

(Clemens: takes correspondence $t \in T \mapsto$ lines through t)

$$\text{CH}_0(\bar{t}) \longrightarrow \text{CH}_0(S)$$

Uses to get rat'l equivalences!

Mumford's Theorem

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Assume $p_g > 0$. Then the following equivalent statements,

- (1) $\exists n$ s.t. $\forall$ 0-cycles A with $\deg(A) \geq n$,
 $A \sim A'$ effective,
- (2) $\exists n$ s.t. the natural map

$$S^n X \times S^n X \rightarrow A_0(X)$$
 $(A, B) \mapsto A - B$ is surjective,
- (3) $\exists n$ s.t. $\forall m$, $A \in S^m X \quad \exists W$ s.t.
 $A \in W \subset S^m X$ of codim $\leq n$ consisting
of points ratly equiv. to A ,
- (4) $\exists C \subset X$ s.t. $J(C) \rightarrow A_0(X)$,

all FAIL.

Proof: (Mumford) Focus on (3): let $A \in S^m X$ = distinct, generic

Let $\omega_0^{(n)}$ be the holomorphic form on $S^n X_{\text{sm}}$.

$\omega_0^{(n)} = \text{sym } \omega$, ω nonzero $\Rightarrow$ generically nondeg.

symplectic form on $X \times X$

$\Rightarrow \omega_0^{(n)}$ is gener. nondeg., i.e. symplectic form.

Step 1: For a family of rationally equivalent eff. cycles

$\alpha: S \rightarrow S^n X$, with some distinct, have

$$\alpha^* \omega_0^{(n)} = 0.$$

Pf idea: basic family of rat. equiv. is just $\mathbb{P}^1 \rightarrow S^n X$.

$\Rightarrow \omega$ must vanish (no form on $\mathbb{P}^1$)

(need whole family of rational equivalences)

So tangent spaces to $S \hookrightarrow S^n X$ must be isotropic

$$\Rightarrow \text{codim} \geq \frac{1}{2} \dim S^n X = n.$$

So, basically "image" of $S^n X \rightarrow A_0(X)$ is $\geq n$ dimension

Philosophy: For Fano surface, $\cdot \lambda^2 H^1(S, \mathcal{O}_S) \cong H^2(S, \mathcal{O}_S) \cong \mathbb{C}$

• H^1 is tangent space to $A_1(X) = \text{Pic}^0(S)$.

• We also just showed that intersection is surjective!

$$\begin{array}{ccc}
 \text{Pic}^0(S) \otimes \text{Pic}^0(S) & \xrightarrow{(\cdot)} & T(S) \\
 \uparrow \text{tangent} \quad \uparrow \text{space} & & \downarrow \text{"bitangent space?"} \\
 H^1(S, \mathcal{O}_S) \otimes H^1(S, \mathcal{O}_S) & \xrightarrow{\cap = \lambda} & H^2(S, \mathcal{O}_S)
 \end{array}$$

Lecture 6.

Appendix: works in characteristic $\neq$ char k

[A]

SLICK PROOF
OF MUMFORD'S.

$H^2_{\text{et}}(X, \mathbb{Q}_\ell)$ transcendental instead of $H^{2,0}$.

We stay / except for lemmas, where $k = \bar{k}$.

Lemma A1. X smooth / k , Y any k -variety.

Write $K = k(Y)$. Then

$$CH^n(X \times_k K) \cong \lim_{\substack{\cup \\ \text{open}}} CH^n(X \times_k U)$$

(THINK OF CYCLES ON $X \times K$ AS FAMILIES OF CYCLES ON $X \times U$)

Proof: First remarks that as sets,

$$(X_K)^{\dim} = \bigcap (X \times_k U) \quad (\text{in fact } \lim)$$

= {points mapping to generic point of Y .}

Can check using functor of points & $K = \varinjlim A_U$,

for codimensions choose transcendence basis.

For $\{U \subset V\}$ have filtered direct system

$$CH^n(X_K) \cong \text{Coker} \left(\bigoplus_{x \in (X_K)^{n-1}} K(x)^* \longrightarrow \bigoplus_{x \in (X_K)^n} \mathbb{Z} \right)$$

$\uparrow \lim$

$$CH^n(X \times U) \cong \text{Coker} \left(\bigoplus_{y \in (X \times U)^{n-1}} k(y)^* \longrightarrow \bigoplus_{y \in (X \times U)^n} \mathbb{Z} \right)$$

delete points!

$$\text{Coker} \left(\bigoplus_{y \in (X \times U)^{n-1}} k(y)^* \xrightarrow{\text{div}} \bigoplus_{(X \times U)^n} \mathbb{Z} \right)$$

$\Rightarrow$ Cokernel commutes with $\uparrow$ filtered direct limit, which is top row.

(note: for $x \mapsto \eta_y$, have $k \hookrightarrow k(x)$
 $\Rightarrow K(x) = k(x)$)

• Lemma A3. Let $k \subset K \subset K'$ be any extensions of $k = \bar{k}$ ^{why? [A]}
 Then the kernel of $\alpha: CH^2(X_K) \xrightarrow[\text{pullback}]{\text{flat}} CH^2(X_{K'})$ is torsion.

Proof: first assume $[K':K] < \infty$. Then this is factored by pullback (i.e. norm) map $\Rightarrow \alpha \times \alpha^* = n \cdot \text{Id}$
 $\Rightarrow$ kernel is n -torsion.
 Can take limits to get for any algebraic extension.

Since we just want to show ker = torsion, can take K, K' alg. closed; $k \subset K \subseteq K'$ and we'll show kernel of α is torsion by composition, so true at K' $\Rightarrow$ at K .
 $\hat{\bigcap}_K \bigcap_{K'} \hat{\bigcap}_K$
 0 is torsion, so $\square$.

• We can write K' ^{now alg. closed} as a limit of finite trans. deg. extensions $K'' \subseteq K'$, for which we have $K'' = K(Y)$ for some k -variety Y . By L1, we then have

$$CH^n(X \times K'') = \varinjlim CH^n(X \times U).$$

But since K is not alg. closed, U has a K -rational point p and we may factor the identity

$$\text{Id}_X: CH^n(X_K) \rightarrow CH^n(X_U) \rightarrow CH^n(X \times \{p\}),$$

showing that the first map is injective. So in fact $CH^n(X_K) \hookrightarrow CH^n(X_{K''})$ is as well.

Taking the limit of $K'' \subset K'$, get the claim. $\square$

• i.e. ignoring torsion, get X defined over $\bar{\mathbb{Q}}$,

$$CH^n(X_{\bar{\mathbb{Q}}})_{\mathbb{Q}} \xrightarrow{\sim} CH^n(X_{\mathbb{C}})_{\mathbb{Q}}.$$

.. Proposition: (Block-Simur decomposition of Δ) |A3

Assume $A_0(X)$ is finite-dimensional, i.e.

$\exists C' \subset X$ s.t. $J(C') \twoheadrightarrow A_0(X)$. Then there exist several subschemes $C', C'' \subset X$, and a 2-cycle P supp'd on $(C' \times X) \cup (X \times C'')$ s.t.

$$n\Delta \sim TP, \text{ some } n.$$

Proof: Let $k \subset \mathbb{C}$ be an alg. closed subfield of finite transcendence degree / $\overline{\mathbb{Q}}$, s.t. C' and X are defined over k . Let $K = k(X)$, and also embed $k \subset K \subset \mathbb{C}$.

We are given that $CH^2((X-C')_{\mathbb{C}}) = 0$, by standard exact sequence

By Lemma 3, this means $CH^2((X-C')_K)$ is torsion, so let N s.t. $N \cdot P = 0$, $P =$ image of diagonal under $CH^2(X \times X) \rightarrow CH^2(X_K)$.

But then by the first Lemma, there exists $U \subset X$ open s.t. $N \cdot \Delta \sim 0$ on $(X-C') \times_k U$.

Wlog take $U = X - C''$, C'' dim ≤ 1 subscheme.

We then have exact sequence

$$\left\{ \begin{array}{l} \text{cycles supp'd on} \\ C' \times X \cup X \times C'' \end{array} \right\} \longrightarrow CH^2(X \times X).$$

$$\longrightarrow CH^2((X-C') \times (X-C'')) \longrightarrow 0,$$

so $\text{image}(NP) = 0 \Rightarrow$ the claim. $\square$

Corollary: $[\Delta]_* = 0 = Id_*$ on $H^{2,0}(X)$. Therefore $Pg = 0$.

Can do some factoring through C', C'' and use

$$H^{2,0}(Curve) = 0 \text{ to kill } [\Delta]_*.$$

.. Bloch's conjecture: $P_g = 0 \Rightarrow T(X) = 0$.

○ Holds for non-general-type. (Voisin's book),
and for Godeaux surfaces like the one we did.

• Can make descending filtration of F :

$$F = CH_0(X) \supseteq A_0(X) \supseteq T(X) \supseteq ?$$

$$\Rightarrow \text{gr}^{\mathbb{Z}} F(CH_0(X)) = \mathbb{Z} \oplus_{\text{deg}} Alb(X) \oplus T(X)$$

~~Adapted~~ An n -dim'l correspondence $\begin{matrix} \overline{T}_X \\ X \longrightarrow X' \end{matrix}$
respects the grading.

Conjecture: For surfaces, $\text{gr} \overline{T}_X$ depends only on
homology class of T .

○ $\Rightarrow$ Bloch's (similar to appendix argument, components of Δ
+ moving).
metaconjecture was dumb.